Generalized phase transitions in finite coupled map lattices

Michael Blank

Russian Ac. of Sci., Inst. for Information Transmission Problems,
Ermolovoy Str. 19, 101447, Moscow, Russia, blank@obs-nice.fr

20.09.1995

Abstract – We investigate generalized phase transitions of type localization - delocalization from one to several Sinai-Bowen-Ruelle invariant measures in finite networks of chaotic elements (coupled map lattices) with general graphs of connections in the limit of weak coupling.

1 Introduction

During last decade a new class of models, so called networks of chaotic elements or coupled map lattices (CML), has been introduced to investigate complex dynamical phenomena in spatially extended systems. A review of various applications of CML one can find in [8]. Mathematical investigation of ergodic properties of such systems was started using methods and ideas of statistical physics in [7] and then was continued in [13, 2, 9, 6, 3]. The main result obtained in these papers was the stability of statistical properties of CML in the limit of weak coupling. In this paper we discuss ergodic properties of finite CML with general graphs of connections among elements in the limit of weak coupling and show that in the general case there may be generalized phase transitions in such systems of type localization - delocalization. These phase transitions correspond to a situation, when trajectories, which should normally be dense, remain confined to a small region (which vanishes, when the coupling constant goes to zero). We call this "localization phenomenon". The first observation of this type was published in [2] (see also [10] for numerical studies of other phase transitions in the simplest dyadic models of CML). We shall give several general statements about Sinai-Bowen-Ruelle (SBR) measures of such systems and then we shall discuss in detail two particular families of CML.

It is worthwhile to discuss the nature of the localization phenomenon. Suppose for a moment that our uncoupled system is a hyperbolic map. Consider a periodic trajectory of a map, in a small neighborhood of which the map is strictly hyperbolic. Then near any point of this trajectory local stable and unstable manifolds are going arbitrary close one to another. Therefore stable and unstable directions can be mixed by means of arbitrary small perturbations, and the result depends on whether the contraction is stronger than expansion (localization), or not. CMLs that we consider are not hyperbolic, but the same result gives the absence of the local expanding property. This property means that a map expands distances between any close enough points. This property distinguishes the situations in Theorems 1.1 and 1.2. To show that this phenomenon is not something obscure, specific for only discontinuous maps, we shall prove its presence for the well known family of quadratic maps in Section 5.

Let $\{f_i\}$ be a sequence of one-dimensional nonsingular mappings $f_i : X \to X = [0,1]$ from the unit interval into itself, let X^d be a direct product of these intervals, and let us denote a point in this direct product by $\bar{x} = (x_1, \ldots, x_d) \in X^d$. Remark that the dimension d needs not to be finite.

Definition 1.1 *By a* coupled map lattice *(CML) we shall mean a map $F_\varepsilon : X^d \to X^d$ defined as follows:*

$$F_\varepsilon(\bar{x}) = \Phi_\varepsilon \circ F(\bar{x}), \tag{1.1}$$

where the map F is a direct product of the maps f_i (i.e. $(F(\bar{x}))_i := f_i(x_i)$), and the map $\Phi_\varepsilon : X^d \to X^d$, describing the coupling, is defined as follows:

$$(\Phi_\varepsilon \bar{x})_i = (1-\varepsilon)x_i + \varepsilon \sum_j \gamma_{ij} x_j, \tag{1.2}$$

and the matrix $\Gamma = (\gamma_{ij})$ is a stochastic matrix, describing the graph of interactions among the maps.

Remark. The matrix Γ needs not to be symmetric, which means that the connections (couplings) may be oriented. We shall consider only connected graphs, however there may be "free" nodes. We shall call a node i of the graph a free node, if $\gamma_{ij} = 0$ for any j (this does not contradict to the connectivity of the graph Γ).

The simplest known type of chaotic maps is so called piecewise expanding (PE) maps.

Definition 1.2 *We shall say that a map f is* piecewise expanding *(PE) if there exists a partition of the interval X into disjoint intervals $\{X_j\}$, such that $f\big|_{\mathrm{Clos}(X_j)}$ is a C^2-diffeomorphism (from the closed interval $\mathrm{Clos}(X_j)$ to its image), and the* expanding constant *of the map*

$$\lambda_f := \inf_{j,\, x \in X_j} |f'(x)|,$$

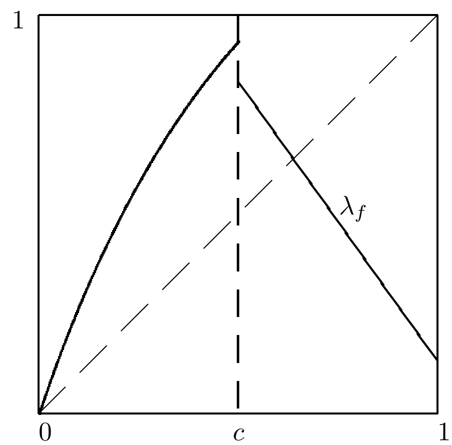

Figure 1: A “typical” PE map.

is positive and for some integer κ the iteration f^κ of the map f has the expanding constant λ_{f^κ} strictly larger than 1.

A typical example of a PE map is shown on Figure 1. So the map even needs not to be continuous. As it is well known, starting from the paper of Lasota & Yorke [12], these maps have all the statistical properties that one can demand on deterministic dynamical system. It has smooth (absolutely continuous) invariant measure μ_f (Sinai-Bowen-Ruelle measure of this map), exponential correlation decay and CLT with respect to this measure, etc. In this paper we want to discuss stability of these properties with respect to small perturbations due to the coupling. We shall emphasize some aspects of this problem, because they seem quite counterintuitive, at least from our point of view.

Definition 1.3 *An* image *of a measure μ under the action of a map f is a new measure $f\mu$, such that $f\mu(A) = \mu(f^{-1}A)$ for any measurable set A.*

Definition 1.4 *Let there exists an open set U in the phase space such that for any smooth measure μ with the support in this set its images $f^n\mu$ converge weakly to a measure μ_f, do not depending on the choice of the initial measure μ. Then the measure μ_f is called a* Sinai-Bowen-Ruelle (SBR) *measure of the map f.*

Definition 1.5 *By a* singular point *of a map f we shall mean a point, where the derivative of the map is not well defined. The set of singular points we shall denote by* Sing$[f]$. *(On Figure 1 the point c is a singular point.)*

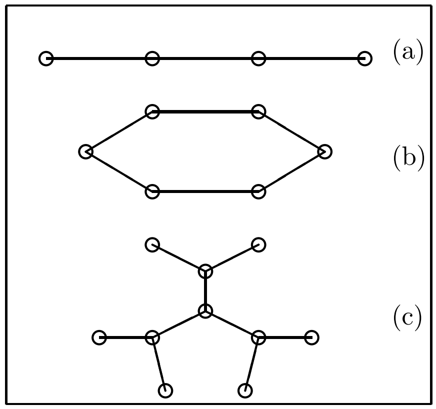

Figure 2: “Typical” graphs Γ: (a) linear chain, (b) cyclic chain, (c) Cayley tree.

Typical examples of the graphs Γ are shown on Figure 2: - linear chain, cyclic chain, Cayley tree.

Let us fix the sequence of maps f_i and the matrix Γ. Clearly for $\varepsilon > 0$ small enough the map F_ε is a piecewise expanding d-dimensional map. The only difference between the multidimensional case and the 1-dimensional one is that one should use the smallest eigenvalue of Jacobi matrix F' of a map F as the expanding constant of the map (see [1] for details). If we assume also that $\lambda_{f_i} > \lambda$, which is large enough (actually much larger than 1), then all the ergodic properties of the corresponding CML follow from general theory of multidimensional PE maps [1]. However, we know only that the expanding constants are positive and larger than 1 for some iteration of the map. Therefore general statements could not be applied here and one should use the specific structure of the map F_ε. In the sequel we shall suppose that each map f_i has only one SBR measure μ_i.

Theorem 1.1 *[2, 9] Let $\lambda_{f_i} \geq \lambda > 2$ for any integer i. Then for any ε small enough CML F_ε has a smooth invariant SBR measure μ_ε converging weakly as $\varepsilon \to 0$ to the direct product of one-dimensional smooth SBR measures μ_i.*

Corollary 1.6 *Let the assumptions of Theorem 1.1 are valid. Then for ε small enough CML F_ε also has the only one SBR measure.*

The main results of this paper are the following statements, generalizing Theorem 1.1 and showing that at least some additional assumptions are necessary for the stability of the direct product of SBR measures μ_i.

Theorem 1.2 *Let for any i the expanding constants of PE maps f_i are strictly positive $\lambda_{f_i} > \lambda > 0$, and there are no periodic singular points. Then for any $\varepsilon > 0$ small enough the CML F_ε has a smooth invariant SBR measure μ_ε converging weakly as $\varepsilon \to 0$ to the direct product of one-dimensional smooth SBR measures μ_i.*

So the only topological obstacle for the stability of statistical properties of CML with respect to weak coupling is the existence of periodic singular points.

Theorem 1.3 *There exists a sequence of PE maps f_i with $\lambda_{f_i} > 1$ and with periodic singular points such that for any $\varepsilon > 0$ small enough CML F_ε has several smooth invariant SBR measures, converging to singular invariant measures of the map F as $\varepsilon \to 0$.*

Remark 1.7 *Under the assumptions of Theorem 1.3 for any $\varepsilon > 0$ small enough Lebesgue measure of the support of every ergodic SBR measure of CML F_ε is of order ε^d.*

This theorem not only states the localization of invariant measures (supports on small sets), but also guarantees their smoothness and the absence of other SBR measures. Actually the localization phenomenon was firstly shown in [2, 3], where it was proved that it is possible to construct CML such that its SBR measure is localized in a small neighborhood of a fixed point, but the statement about the smoothness of invariant measures, absence of other SBR measures, and investigation of their properties are new. It is worthwhile to remark that we assumed in Theorem 1.3, that all the expanding constants λ_{f_i} are greater than one, which is not the general case for CML, constructed of Lasota-Yorke type maps. Therefore even more strong localization may take place, when there are no smooth SBR measures for CML for any weak enough coupling.

Theorem 1.4 *There exists a sequence of PE maps f_i with periodic singular points such that for any $\varepsilon > 0$ small enough CML F_ε has only singular invariant SBR measures, converging to singular invariant measures of the map F as $\varepsilon \to 0$.*

This statement shows that a system of chaotic maps can be stabilized by arbitrary weak coupling. We shall show further that the opposite statement is also true, i.e. a system of stable maps can become a chaotic CML under arbitrary weak coupling.

These results may be considered as some kind of generalized phase transitions, because when the coupling strength ε goes to zero we can have two quite different types of behavior - there are two different "phases" - with one SBR measure and with several of them.

A particular case of Theorem 1.2 for cyclic graph Γ and identical maps f_i with expanding constants strictly greater than 1 was proved earlier in a different way in [11].

2 Functions of generalized bounded variation

We mainly follow here [1, 2, 3]. Let $X^d = [0,1]^d \subset \mathbb{R}^d$ (or d-dimensional unit torus) with the uniform norm $|x| = \max\{|x_i| : x \in \mathbb{R}^d, i = 1,\ldots,d\}$ and d-dimensional Lebesgue measure $m = m_d$. Let us introduce the following notation:

$$\mu(Y,h) = \int_Y h(x)d\mu(x); \quad \mu(Y) = \mu(Y,\mathbf{1});$$

$$\mathrm{Osc}(h,Y,x) = \sup\{|h(x) - h(y)| : y \in Y\};$$

$$\mathrm{Osc}(h,Y) = \sup\{\,\mathrm{Osc}(h,Y,x) : x \in Y\};$$

$$W(h,Y,t) = m(Y,\, \mathrm{Osc}(h, B_t(x) \cap Y, x));$$

$$B_t(x) = \{y \in X^d : |x-y| \le t\};$$

$$\mathrm{var}(h,Y) = \frac{1}{d}\limsup_{t\to 0}\left(\frac{1}{t}\inf_{\hat{h}\in h\subset\mathbf{L}} W(\hat{h},Y,t)\right)$$

$$\mathrm{var}(h) = \,\mathrm{var}(h,X^d), \quad ||h||_v = \,\mathrm{var}(h) + ||h||, \quad ||h|| = m(X^d,|h|).$$

Definition 2.1 *The functional* $\mathrm{var}(h,Y)$ *we shall call the generalized* variation *(or simply variation) of the function h over the set $Y \subseteq X^d$ and the functional $Osc(h,Y)$ shall call* oscillation. *The Banach space of integrable functions with bounded variation*

$$\{h : X^d \to \mathbb{R}^1 \;:\; \mathrm{var}(h) < \infty\},$$

equipped with the variational norm

$$||h||_v = \,\mathrm{var}(h) + ||h||$$

we shall denote by $\mathbf{BV}(X)$ *(or simply by* $\mathbf{BV}$*) and shall call the* space of functions of bounded variation.

The following statement provides some elementary facts about the the properties of these functionals.

Lemma 2.2 *Let functions h, h_1, h_2 be of bounded variation and let Y, Y_1, Y_2 be connected closed subsets of X^d. Then the following inequalities are valid:*

1. $\mathrm{var}(h_1 + h_2, Y) \le \,\mathrm{var}(h_1,Y) + var(h_2,Y)$.
2. $\mathrm{var}(h, Y_1 \cup Y_2) \le \,\mathrm{var}(h,Y_1) + \,\mathrm{var}(h,Y_2)$ *and this inequality becomes an equality if* $\mathrm{Clos}(Y_1) \cap \mathrm{Clos}(Y_2) = \emptyset$.
3. $\mathrm{var}(ch,Y) = c\,\mathrm{var}(h,Y)$ *for any nonnegative number c.*
4. $\mathrm{var}(h_1 \times h_2, Y) \le \,\mathrm{var}(h_1,Y)\,||h_2||_\infty + \,\mathrm{var}(h_2,Y)\,||h_1||_\infty$.

5. $\mathrm{Osc}(h,Y) \le \mathrm{var}(h,Y)$.

6. for any $x \in Y$

$$|h(x)| \le \mathrm{var}(h,Y) + \frac{1}{m(Y)} \int_Y |h(y)|\,dy).$$

7. let $Y=[a,b]$ *then*

$$|h(a)+h(b)| \le \mathrm{var}(h,Y) + \frac{2}{m(Y)} \int_Y |h(x)|\,dx. \tag{2.1}$$

8. for any numbers $0 \le a_1 < a_2 < \ldots < a_n \le 1$

$$\inf_{\hat{h}\in h\subset \mathbf{L}} \sum_i |\hat{h}(a_i) - \hat{h}(a_{i+1}| \le \mathrm{var}(h,\,[a_1,a_n]).$$

Proof. All the properties here are quite the same as for usual one-dimensional variation. However to show how this technics works, let us prove the only nontrivial items (5) - (8). Suppose that the set Y is a segment. Let us fix $0<t \ll 1$ and choose an integer $k>0$ such that

$$(k-1)t < m(Y) \le kt.$$

Then

$$t \ge m(Y)/k > t \times m(Y)/(t+m(Y)).$$

Consider a partition of the segment Y to consecutive disjoint open intervals Y_i of length $m(Y)/k$, such that the union of their closures contains Y. Then

$$W(h,Y,t) = m(Y,\,\mathrm{Osc}(h,B_t(x)\cap Y,x)) \ge m(Y, \sum_i \mathrm{Osc}(h,Y_i,x)\,\mathbf{1}_{Y_i}(x))$$

$$= \sum_i m(Y_i)\;\mathrm{Osc}(h,Y_i) = \frac{m(Y)}{k}\;\mathrm{Osc}(h,Y)$$

$$> t\frac{m(Y)}{t+m(Y)}\;\mathrm{Osc}(h,Y)$$

Hence for every $0<t<<1$

$$\mathrm{Osc}(h,Y) < \frac{t+m(Y)}{m(Y)}\frac{1}{t} W(h,Y,t)$$

Now going to the limit as $t \to 0$ we obtain the required inequality for the case of the connected Y. The general case, when the set Y consists of several connected components is reduced to the considered one by the restriction of the function h to each of the connected components of Y. The item (5) is proved.

The property (6) is a simple consequence of (5), because clearly the value of the function could be estimated by the sum of its oscillation and its means value.

To prove the item (7) set $c = (a+b)/2$, $e = (b-a)/2$, $E = [0, e]$ and consider a function $H : E \to R^1$, defined via

$$H(x) = |h(c-x)| + |h(c+x)|, \quad x \in E.$$

By the item (5)

$$\begin{aligned}\mathrm{Osc}(H, E) &\leq \mathrm{var}(H, E) \\ &\leq \mathrm{var}(h, [a, c]) + \mathrm{var}(h, [c, b]) \leq \mathrm{var}(h, Y).\end{aligned}$$

Besides, by the definition of the oscillation we have for all $x \in E$

$$\begin{aligned}H(x) &\leq \mathrm{Osc}(H, E) + \frac{2}{b-a} \int_0^e H(x)\,dx \\ &\leq \mathrm{var}(h, Y) + \frac{2}{m(Y)} \int_Y |h(x)|\,dx,\end{aligned}$$

because the second addend is equal to the mean value of the function $|h|$. Thus, setting $x = e$ and using that $H(e) = |h(a) + h(b)|$, we obtain the required inequality. The item (7) is proved.

It remains to prove the item (8).

$$\begin{aligned}\sum_i |\hat{h}(a_i) - \hat{h}(a_{i+1}| &\leq \sum_i \mathrm{Osc}(h,\ [a_i, a_{i+1}]) \\ &\leq \sum_i \mathrm{var}(h,\ [a_i, a_{i+1}]) \leq \mathrm{var}(h,\ [a_1, a_n])\end{aligned}$$

∎

Remark, that the property (8) shows the correspondence between our generalized variation and the usual one-dimensional variation.

We shall need also a multidimensional analogue of the inequality (2.1).

Lemma 2.3 *[2, 3] Let $I^d \subseteq X$ be a direct product of d intervals: $I^d = I_1 \times \ldots \times I_d$ (i.e. I^d is a rectangle). Then*

$$\mathrm{var}(h\,\mathbf{1}_{I^d}) \leq 2\,\mathrm{var}(h, I^d) + \frac{2}{\min_j |I_j|} \int_X |h|\,dm_d \tag{2.2}$$

for any function $h : X \to \mathbb{R}^1$ of bounded variation.

This statement follows from the estimate of the trace of a function on a boundary δI^d of the rectangle I^d:

$$\int_{\delta I^d} |h| \, dm_{d-1} \leq \operatorname{var}(h, I^d) + \frac{2}{\min_j |I_j|} \int_X |h| \, dm_d, \tag{2.3}$$

where m_{d-1} is $(d-1)$-dimensional Lebesgue measure on the surface δI^d. It is worthwhile to remark that estimates of this type can be obtained for arbitrary domains $Y \subseteq X$:

$$\operatorname{var}(h \, \mathbf{1}_Y) \leq A(Y) \operatorname{var}(h, Y) + B(Y) \int_X |h| \, dm_d,$$

but the coefficients $A(Y)$ and $B(Y)$ depend crucially on the form of the domain and can be arbitrary large even for small domains (actually they even need not to be finite in the general case).

3 Operator approach for CML. Generic case.

Operator approach for CML is based on the investigation of the Perron-Frobenius operator (transfer-matrix) $\mathbf{P}_{F_\varepsilon}$ describing the action of the CML F_ε on densities of smooth measures on X^d. For $\varepsilon \geq 0$ small enough this operator leaves the space of functions of bounded variation invariant, and the main idea, which was firstly proposed in [12] for the investigation of one-dimensional PE maps, is to obtain for arbitrary integers k estimates of the following type:

$$\operatorname{var}(\mathbf{P}_{F_\varepsilon}^k h) \leq C\alpha^k \operatorname{var}(h) + \beta ||h||, \tag{3.1}$$

where $0 < \alpha < 1$ and $C, \beta < \infty$. Remind, that for the uncoupled map F ($\varepsilon = 0$), the analogous inequality:

$$\operatorname{var}(\mathbf{P}_F^k h) \leq C_0 \alpha_0^k \operatorname{var}(h) + \beta_0 ||h||, \tag{3.2}$$

immediately follows from the well known properties of the Perron-Frobenius operators for the one-dimensional PE maps f_i. Now if the operator $\mathbf{P}_{F_\varepsilon}$ satisfies the inequality (3.1), then a standard technics, based on abstract ergodic theorem due to Ionescu-Tulchea and Marinescu (see the suitable for this approach variant in [1]) gives a possibility to obtain the spectral decomposition of this operator as a sum of a contraction and a finite dimensional projector and to obtain all the standard statistical properties.

Remark 3.1 *The class of coupling maps Φ_ε should be defined up to a nonsingular (piecewise smooth) conjugation.*

Indeed, suppose that $\tilde{f} := \varphi f \varphi^{-1}$, $y = \varphi x$ and $x \to \Phi_\varepsilon \circ F x$ then

$$y \to (\varphi \circ \Phi_\varepsilon \circ \varphi^{-1}) \circ \tilde{F},$$

where by $\tilde{F}$ we mean a direct product of the conjugated maps $\tilde{f}_i$:

$$\tilde{F} := \varphi \circ F \circ \varphi^{-1}.$$

Let us start from the situation without the periodic singular points. Consider the Perron-Frobenius operator, corresponding to CML:

$$\mathbf{P}_{F_\varepsilon} = \mathbf{Q}_\varepsilon \mathbf{P}_F,$$

where the operator $\mathbf{Q}_\varepsilon$ corresponds to the coupling, and $\mathbf{P}_F$ is the Perron-Frobenius operator for the uncoupled system. Suppose for a moment that these two operators are commutative. Then

$$\mathbf{P}^k_{F_\varepsilon} = \mathbf{Q}^k_\varepsilon \mathbf{P}^k_F.$$

From the definition of CML it follows that the coupling operator $\mathbf{Q}_\varepsilon$ is close to identity in the following strong sense:

$$||\mathbf{Q}_\varepsilon||_v \leq 1 + B\varepsilon, \tag{3.3}$$

where the constant $B < \infty$ does not depend on the parameter $\varepsilon > 0$ for ε small enough. Therefore

$$||\mathbf{P}^k_{F_\varepsilon}||_v \leq ||\mathbf{Q}^k_\varepsilon h||_v ||\mathbf{P}^k_F h||_v \leq C(1 + B\varepsilon)^k \alpha_0^k ||h||_v + (1 + B\varepsilon)^k \beta_0 ||h||. \tag{3.4}$$

And thus, choosing k large enough, such that

$$\alpha := C(1 + B\varepsilon)^k \alpha_0^k < 1,$$

we shall have the main inequality.

Unfortunately, these two operators are not commutative in the general case. Therefore to apply this idea we shall construct a new operator $\tilde{\mathbf{Q}}_\varepsilon$, such that

$$\mathbf{P}_{\tilde{F}} \tilde{\mathbf{Q}}_\varepsilon = \mathbf{Q}_\varepsilon \mathbf{P}_F,$$

to use the operator $\tilde{\mathbf{Q}}_\varepsilon$ instead of $\mathbf{Q}_\varepsilon$.

Consider a partition of the phase space X^d into rectangles Δ_j, such that the restriction of the map F to Δ_j is a diffeomorphism (i.e. every Δ_j is a direct product of some intervals of monotonicity for the maps f_i). Then

$$\mathbf{P}_F h = \sum_j \mathbf{P}_{F,j} h, \tag{3.5}$$

where

$$\mathbf{P}_{F,j}h(x) := h(F_j^{-1}x)\,|\det(F_j^{-1}x)'|\,\mathbf{1}_{F\Delta_j}(x), \tag{3.6}$$

$F_j := F\big|_{\Delta_j}$, and $(Fx)'$ is the Jacobi matrix of the map F.

Then on a small rectangular extension $\tilde{\Delta}_j$ of the rectangle Δ_j the operator $\tilde{\mathbf{Q}}_\varepsilon$ is defined as follows:

$$\tilde{\mathbf{Q}}_\varepsilon h_{\tilde{\Delta}_j} = \mathbf{P}_{\tilde{F}_j^{-1}}(\mathbf{Q}_\varepsilon \mathbf{P}_F h_{\Delta_j}), \tag{3.7}$$

where h_{Δ_j} is the restriction of a function h to the set Δ_j, and $\tilde{F}_j$ is a C^2-smooth extension to a diffeomorphism of the map F_j on $\tilde{\Delta}_j$. The collection of maps $\tilde{F}_j$ on the overlapping rectangles $\tilde{\Delta}_j$ may be considered as a multivalued map $\tilde{F}$ from the unit cube into itself. Denote the restriction of the operator $\tilde{\mathbf{Q}}_\varepsilon$ to the j-th branch of the map by $\tilde{\mathbf{Q}}_{\varepsilon,j}$, and let us define a function $J(x) := |\det(\tilde{F}_j^{-1}x)'|$. Then

$$\tilde{\mathbf{Q}}_{\varepsilon,j}h(x) = J(x)\mathbf{Q}_\varepsilon \mathbf{P}_F h(\tilde{F}x)$$

If we prove now that the operator $\tilde{\mathbf{Q}}_\varepsilon$ is also close to identity (i.e. we prove an inequality of type (3.3) with may be another constant, say B', which also will not depend on ε), then the argument above can be applied in this case. To prove this inequality we mainly follow the idea introduced in our paper [4] in the investigation of random perturbations of one-dimensional PE maps. Therefore here we shall discuss in detail only the first step of the estimation, where the difference between the multidimensional case (considered in this paper) and one-dimensional (considered in [4]) really takes place.

Let us fix some integer j and consider only the branch F_j on the rectangle Δ_j. For the sake of simplicity we change the notation here and drop the index j, i.e. we shall denote F_j by F and Δ_j by Δ. Let for some point $y \in \Delta$ we have

$$\operatorname{var}\left(\frac{J}{J(y)}\right), \ \operatorname{var}\left(\frac{J(y)}{J}\right) \le \beta \ ,$$

and let $\mathbf{BV}_0 := \{h \in \mathbf{BV}(X), \ h(u) = 0 : \ u \in \delta X^d\}$. Then $||h|||_\infty \le \operatorname{var}(h)$ for all $h \in \mathbf{BV}_0$.

Lemma 3.2 *Let $\mathbf{Q}$ be a (sub)Markov operator satisfying*

$$\operatorname{var}(\mathbf{Q}h) \le \operatorname{var}(h) + C \cdot ||h||$$

for all $h \in \mathbf{BV}_0$. Let

$$\tilde{\mathbf{Q}} := \mathbf{P}_{\tilde{F}_i^{-1}}\mathbf{Q}\mathbf{P}_{F_i},$$

then $\mathbf{P}_{\tilde{F}}\tilde{\mathbf{Q}} = \mathbf{Q}\mathbf{P}_F$ *and*

$$\mathrm{var}(\tilde{\mathbf{Q}}h) \leq (1+\beta)^2 \cdot \, \mathrm{var}(h,\Delta) + C(1+\beta)||J||_\infty \cdot ||h \cdot \mathbf{1}_\Delta||$$

for all $h \in \mathbf{BV}_0$.

Proof. Let $g(x) := \frac{J(x)}{J(y)}$. Then $\mathrm{var}(g) \leq \beta$, $||g||_\infty \leq 1 + \frac{\beta}{2}$ and

$$\tilde{Q}h(x) = g(x) \cdot \mathbf{Q}\mathbf{P}_F h(J(y) \cdot h)(\tilde{F}x)$$

because $\mathbf{Q}\mathbf{P}_F$ is a linear operator. Therefore

$$\begin{aligned}
\mathrm{var}(\tilde{\mathbf{Q}}h) &\leq \mathrm{var}(g) \cdot ||\mathbf{Q}\mathbf{P}_F(J(y) \cdot h)||_\infty + ||g||_\infty \cdot \, \mathrm{var}(\mathbf{Q}\mathbf{P}_F(J(y) \cdot h)) \\
&\leq \beta \cdot \frac{1}{2}\,\mathrm{var}(\mathbf{Q}\mathbf{P}_F(J(y) \cdot h)) + (1+\frac{\beta}{2}) \cdot \, \mathrm{var}(\mathbf{Q}\mathbf{P}_F(J(y) \cdot h)) \\
&\leq (1+\beta) \cdot (\, \mathrm{var}(\mathbf{P}_F(J(y) \cdot h)) + C \cdot ||\mathbf{P}_F(J(y) \cdot h)||) \\
&\leq (1+\beta) \cdot \left(\mathrm{var}((\frac{J(y)}{J} \cdot h) \circ F^{-1} \cdot \mathbf{1}_{F\Delta}) + C \cdot ||J(y) \cdot h \cdot \mathbf{1}_\Delta|| \right) \\
&\leq (1+\beta) \cdot \left(\mathrm{var}(\frac{J(y)}{J} \cdot h \cdot \mathbf{1}_\Delta) + C \cdot ||J||_\infty \cdot ||h \cdot \mathbf{1}_\Delta|| \right) \\
&\leq (1+\beta) \cdot \left(\mathrm{var}(\frac{J(y)}{J})||h \cdot \mathbf{1}_\Delta||_\infty + |\frac{J(y)}{J}\|_\infty \,\mathrm{var}(h \cdot \mathbf{1}_\Delta) + C \cdot ||J||_\infty \cdot ||h \cdot \mathbf{1}_\Delta|| \right) \\
&\leq (1+\beta) \cdot \left(\beta \cdot \frac{1}{2}\,\mathrm{var}(h \cdot \mathbf{1}_\Delta) + (1+\frac{\beta}{2}) \cdot \, \mathrm{var}(h \cdot \mathbf{1}_\Delta) + C \cdot ||J||_\infty \cdot ||h \cdot \mathbf{1}_\Delta|| \right) \\
&\leq (1+\beta)^2\,\mathrm{var}(h \cdot \mathbf{1}_\Delta) + (1+\beta)C||J||_\infty \cdot ||h \cdot \mathbf{1}_\Delta||
\end{aligned}$$

■

Remark 3.3 *To obtain an arbitrary small value of* β *above it is enough to choose fine enough subpartitions for our maps* f_i.

This finishes the proof of Theorem 1.2.

It seems, that the same idea could be applied for a general multidimensional piecewise expanding map. However, there are two obstacles. The first one is pure topological - existence of periodic singular points, which we shall discuss in the next section. The second obstacle is due to the fact that in the general case elements of the partition for F (and especially for F^n) are not rectangles. As we mentioned before this can lead to large (and even infinite) coefficients in the corresponding estimates for the Perron-Frobenius operator $\mathbf{P}_F$.

4 Operator approach for CML. Localization for the case with periodic singular points.

Consider now the case with periodic singular points. To prove Theorems 1.3 and 1.4 it is enough to construct examples of CML with periodic singular points, satisfying the assumptions of these theorems. We shall show that it is enough to consider two-dimensional case ($d = 2$) with identical maps $f_1 = f_2 = f$, having only two fixed singular points c_1, c_2 and the following symmetrical coupling:

$$c_1 = f(c_1) \neq f(c_2) = c_2, \quad \gamma_{11} = \gamma_{22} = 0, \quad \gamma_{12} = \gamma_{21} = 1.$$

Consider the following two families of maps:

$$f_{b,c}^{(1)}(x) = \begin{cases} \frac{1}{2} - \frac{x/2}{c-c/b}, & \text{if } 0 \le x \le c - c/b; \\ b(x-c) + c, & \text{if } c - c/b \le x \le c; \\ -b(x-c) + c, & \text{if } c \le x \le c + c/b; \\ \frac{x+c+c/b}{1-2c-2c/b}, & \text{if } c + c/b \le x \le 1/2; \\ 1 - f(1-x), & \text{if } 1/2 \le x \le 1. \end{cases}$$

$$f_{b,c}^{(2)}(x) = \begin{cases} -\frac{b-c}{c}x + b, & \text{if } 0 \le x \le c; \\ -\frac{c}{0.5-c}x + 0.5, & \text{if } c \le x \le 1/2; \\ 1 - f(1-x), & \text{if } 1/2 \le x \le 1. \end{cases}$$

$$\lambda_{f_{b,c}^{(1)}} := b > 1, \quad \lambda_{f_{b,c}^{(2)}} := \min\left\{ \frac{c}{\frac{1}{2} - c}, \ \frac{b-c}{c} \right\} < 1.$$

$$\lambda_{(f_{b,c}^{(2)})^2} := \frac{b-c}{\frac{1}{2} - c} > 1$$

Let us start the investigation from the first family $f_{b,c}^{(1)}$. Fix some $1 < b < 2$, $0 < c < 1/2$, $\varepsilon > 0$. Then the expanding constant $\lambda_f = b > 1$. Consider the first two points of the trajectory of the point $(c, 1-c)$ of the CML F_ε constructed by means of this map:

$$\begin{pmatrix} c \\ 1-c \end{pmatrix} \xrightarrow{f,\Phi_\varepsilon} \begin{pmatrix} c + \varepsilon(1-2c) \\ 1 - c - \varepsilon(1-2c) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} c - \varepsilon b(1-2c) \\ 1 - c - \varepsilon b(1-2c) \end{pmatrix}$$

$$\xrightarrow{\Phi_\varepsilon} \begin{pmatrix} c - \varepsilon(b(1-2c) + 2c - 1) + 2\varepsilon^2 b(1-2c) \\ 1 - c + \varepsilon(b(1-2c) + 2c - 1) - 2\varepsilon^2 b(1-2c) \end{pmatrix}$$

Denote now

$$c_- := c - \varepsilon(b-1)(1-2c), \quad c_+ := c + \varepsilon(1-2c)$$

$$K_1 := [c_-, c_+] \times [1 - c_+, 1 - c_-], \quad K_2 := [1 - c_+, 1 - c_-] \times [c_-, c_+]$$

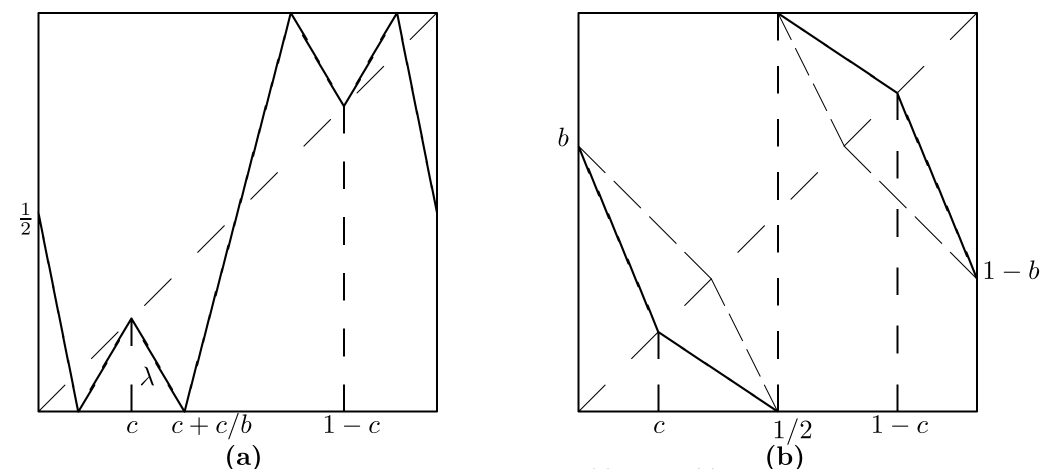

Figure 3: Local maps, (a) $f_{b,c}^{(1)}$, (b) $f_{b,c}^{(2)}$

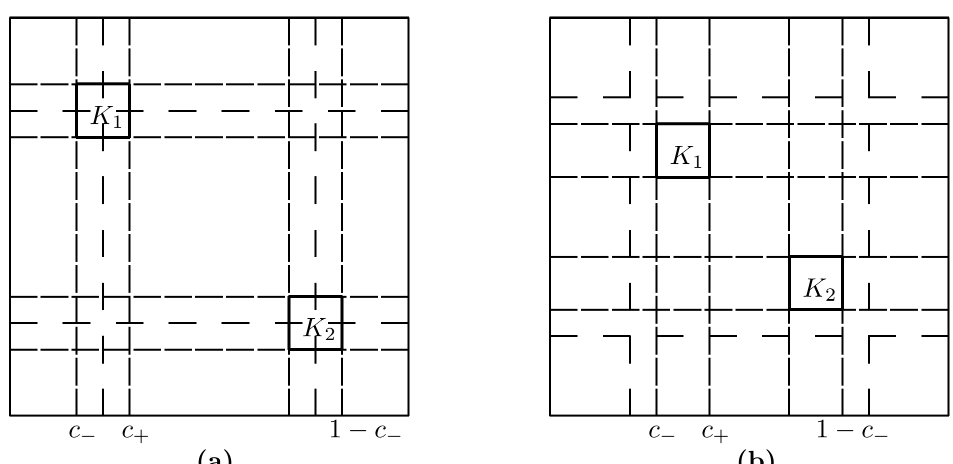

Figure 4: Space localization: (a) $f_{b,c}^{(1)}$, (b) $f_{b,c}^{(2)}$

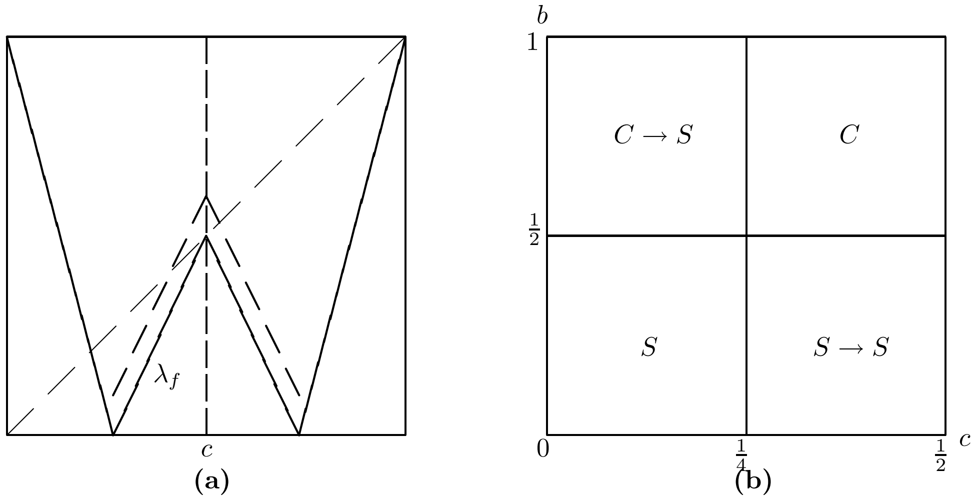

Figure 5: (a) Local action of coupling near fixed singular points, (b) bifurcation diagram for the map $f_{b,c}^{(2)},\ 0 \le b \le 1,\ 0 \le c \le 1/2$

Lemma 4.1 *$F_\varepsilon K_i \subset K_i$ for every $i = 1, 2$.*

Proof. The restriction of the map F_ε to the rectangle K_1 is a continuous map, such that the boundary of the rectangle K_1 is mapped by F_ε into K_1 and at least one inner point of K_1 (the point $(c, 1-c)$) is also mapped by F_ε into K_1. To show this, it is enough to prove that every corner of the rectangle K_1 is mapped into K_1. Let us prove it for the left lower corner $(c_-, 1-c_+)$. Denote $t := \varepsilon(1-2c),\ \beta := b - 1 < 1$. Then $c_- = c - t\beta,\ c_+ = c + t$ and

$$\begin{pmatrix} c_- \\ 1-c_+ \end{pmatrix} = \begin{pmatrix} c - t\beta \\ 1-c-t \end{pmatrix} \xrightarrow{f} \begin{pmatrix} c - bt\beta \\ 1-c+bt \end{pmatrix}$$

$$\xrightarrow{\Phi_\varepsilon} \begin{pmatrix} c - bt\beta + \varepsilon(1-2c+tb+tb\beta) \\ 1-c+tb-\varepsilon(1-2c+tb+tb\beta) \end{pmatrix}$$

$$= \begin{pmatrix} c - tb + tb - bt\beta + \varepsilon(1-2c+tb+tb\beta) \\ 1-c-t+t+tb-\varepsilon(1-2c+tb+tb\beta) \end{pmatrix}$$

$$= \begin{pmatrix} c - tb + tb - bt\beta + \varepsilon(1-2c+tb+tb\beta) \\ 1-c-t+tb(1-\varepsilon(1+\beta)) \end{pmatrix}$$

$$> \begin{pmatrix} c - tb \\ 1-c-t \end{pmatrix} = \begin{pmatrix} c_- \\ 1-c_+ \end{pmatrix},$$

where by $\xi > \eta$ we mean that every coordinate of the vector ξ is greater than the corresponding coordinate of the vector η.

On the other hand,

$$\begin{pmatrix} c - bt\beta + \varepsilon(1-2c+tb+tb\beta) \\ 1-c+tb-\varepsilon(1-2c+tb+tb\beta) \end{pmatrix}$$

$$= \begin{pmatrix} c+t-b\varepsilon((1-2c)\beta - t - t\beta) \\ 1-c+t-\varepsilon((2-b)(1-2c)+tb+tb\beta) \end{pmatrix}$$

$$< \begin{pmatrix} c+t \\ 1-c+t \end{pmatrix} = \begin{pmatrix} c_+ \\ 1-c_- \end{pmatrix}.$$

In the same way one can show that three other corner points are mapped into the rectangle K_1. The proof for the rectangle K_2 is analogous. Therefore these rectangle are mapped into themselves under the action of F_ε. ■

Lemma 4.2 *The restriction of the map F_ε to a rectangle K_1 (K_2) has a smooth invariant measure (SBR measure) μ_1 (μ_2).*

Proof. Denoting $c_1 = c$, $c_2 = 1-c$ we consider the following change of variables:

$$y_i := \frac{x_i - c_i}{\varepsilon}, \tag{4.1}$$

which gives new local maps $\tilde{f}_i$ from the neighborhoods of the points c_i (whose direct product gives the rectangle K_1) into some new intervals with sides of order 1:

$$y_i \to (1-\varepsilon)\tilde{f}(y_i) + C_i + \mathcal{O}(\varepsilon), \tag{4.2}$$

where $\tilde{f}_i(0) = 0$ and $\prod_i C_i \neq 0$. The shape of the new local map $\tilde{f}$ in a neighborhood of a fixed singular point is shown by a dotted line on Figure 5.a. This new system is also a couple map lattice, but without singular periodic points (for $\varepsilon > 0$ small enough). Therefore we can apply here the results of Theorem 1.2 to obtain the existence of smooth invariant measures. ■

Lemma 4.3 *There are only two SBR measures μ_i, $i = 1,2$ of the map F_ε and their supports* $\mathrm{supp}(\mu_i) \subset K_i$.

Proof. Let us construct a new map $\tilde{F}$ from X^2 into itself, which will differ from the map F only on the rectangles K_i:

$$\tilde{F}(x) = \begin{cases} F(x) - 2\varepsilon(1,-1), & \text{if } x \in K_1; \\ F(x) + 2\varepsilon(1,-1), & \text{if } x \in K_2; \\ F(x), & \text{otherwise.} \end{cases}$$

Then for $\varepsilon > 0$ small enough the map $\tilde{F}_\varepsilon := \Phi_\varepsilon \circ \tilde{F}$ is a PE map. The difference between this map and the map F_ε is that, it has no "traps" around singular points c and $1-c$. Now using the same argument as in the proof of Theorem 1.2 one can prove the existence and the uniqueness of a smooth invariant measure $\mu_{\tilde{F}}$ of this map, which will have a positive density on the whole rectangle X^2. Therefore a.a. trajectory of the map $\tilde{F}_\varepsilon$ is dense on X^2.

This shows that a.a. trajectory of the original map F_ε hits eventually into one of the sets K_i, because it coincides with some trajectory of the map $\tilde{F}_\varepsilon$ up to the moment, when it hits into one of the sets K_i. ■

This finishes the proof of Theorem 1.3.

In a close way one can investigate also the second family of maps $f^{(2)}_{b,c}$. We shall use the same notation as for the first family. Fix some $1/2 < b < 1$, $0 < c < 1/2$, $\varepsilon > 0$. Then the expanding constant $\lambda_{f^2} = (b-c)/(1/2-c) > 1$, however $\lambda_f = \min\{(b-c)/c, c/(1/2-c)\}$ may be less than 1. Consider the first two points of the trajectory of the point $(c, 1-c)$ of the CML F_ε constructed by means of this map:

$$\begin{pmatrix} c \\ 1-c \end{pmatrix} \xrightarrow{f,\Phi_\varepsilon} \begin{pmatrix} c+\varepsilon(1-2c) \\ 1-c-\varepsilon(1-2c) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} c-2\varepsilon c \\ 1-c-2\varepsilon c \end{pmatrix}$$

$$\xrightarrow{\Phi_\varepsilon} \begin{pmatrix} c-\varepsilon(4c-1)+4\varepsilon^2 c \\ 1-c+\varepsilon(4c-1)-4\varepsilon^2 c \end{pmatrix}$$

Denote now

$$c_- := c-\varepsilon(4c-1), \quad c_+ := c+\varepsilon(1-2c)$$

$$K_1 := [c_-, c_+] \times [1-c_+, 1-c_-], \quad K_2 := [1-c_+, 1-c_-] \times [c_-, c_+]$$

Lemma 4.4 *Let $4c < 1$, then $F_\varepsilon K_i \subset K_i,\ i = 1,2$.*

The proof of this statement is analogous to the proof of Lemma 4.1.

Lemma 4.5 *Let $4c < 1$, then restriction of the map F_ε to a rectangle K_1 (K_2) has a globally stable fixed point p_1 (p_2).*

Proof. The main difference between the considered situations and the result of Lemma 4.2 is that the restriction of this CML to one of the rectangles K_i is not a PE map, because for any its iterations the expanding constant is strictly less than 1. Indeed, the derivative of the local map is equal to $c/(1/2-c) < 1$ for $4c < 1$, and the coupling can only decrease the expansion. ■

Theorem 1.4 is proved.

Using the same argument as in the proof of Lemmas 4.1, 4.2, one can show that the situation will be quite different when $0 < b < 1/2$. The point is, in this region of parameters the uncoupled system is not expanding and has two locally stable fixed points.

Lemma 4.6 *Let $4c > 1$ and $0 < b < 1/2$, then $F_\varepsilon K_i \subset K_i,\ i = 1, 2$ and CML F_ε has two singular SBR measures with the supports on cycles of period 2.*

The bifurcation picture for the second family of maps is shown on Figure 5.b. The uncoupled system is chaotic when $b > 1/2$ and stable otherwise. We denoted this two possibilities by letters "C" and "S". Consider the coupled system for a small enough $\varepsilon > 0$. Lemmas 4.4, 4.5 show that when $4c < 1$ there is a transition from the chaotic behavior to the stable one ($C \to S$). Lemma 4.6 describes another transition from stable fixed points to stable period 2 cycles ($S \to S$). When $4c < 1,\ 0 < b < 1/2$ or $4c > 1,\ 1/2 < b < 1$ there are no "traps" around the fixed singular points and therefore there are no "phase" transitions.

Generalization of our results to the general case of periodic singular points (instead of fixed points) and arbitrary dimension d is straightforward and it is easy to formulate some simple sufficient conditions of the localization phenomenon. However any CML, whose local map has only one fixed singular point, shows the absence of the localization. Therefore the investigation of necessary conditions is a more complex problem and will be published elsewhere.

5 Localization in CML, constructed by smooth maps

Actually the appearance of the localization in CML is quite unexpected and may seem to be a consequence of the discontinuity of PE maps. To show that it is not so and that this phenomenon is not artificial and is generic, we shall prove in this section its existence for smooth local maps. The simplest way to do it is to smooth the maps $f^{(i)}_{b,c}$ near periodic singular points. Howether we shall consider a more general situation - a well known family of quadratic maps $f_a(x) := ax(1 - x),\ 3 < a \leq 4$. This situation differs from the results of Teorems 1.3,1.4 in that sense, that the localization takes place only for nonzero coupling strength. Let the value of the parameter a is close enough to 4 and the map f_a has a smooth SBR measure. It is well known that the set of such values of the parameter a is of positive Lebesgue measure. We shall consider a CML, constructed by means of 2 identical square maps with the parameter a.

Lemma 5.1 *There exists a constant $3 < a_0 \leq 4$, and an inteval of values of the coupling strength $0.14 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 0.2$ such the the CML F_ε has a stable periodic trajectory with period 2:*

$$F_\varepsilon(p_1, p_2) = (p_2, p_1); \quad F_\varepsilon(p_2, p_1) = (p_1, p_2),$$

where $p_1 < 1/2 < p_2 < 1$ and $f'_a(p_1) \times |f'_a(p_2)| > 1$, and therefore an SBR measure, localized on this trajectory.

Proof. We can follow the same construction as in the case of PE maps to find two "traps" for CML, but in the case of square maps this way is too complex. Therefore we shall use the fact that the map f_a is smooth and smoothly depends on the parameter a. Therefore it is enough to prove the existence of a stable periodic trajectory for a given pair of values (a, ε). Let us set $a = 4$, $\varepsilon = .17$. A simple calculation shows that the pair of points $p_1 = 0.484989\dots$ and $p_2 = 0.893799\dots$ defines the periodic trajectory with period 2 for this pair of values (a, ε). To prove its stability we shall calculate Jacobian matrices of the CML in this poits and shall show that eigenvalues of their product are positive and less than 1. The Jacobian matrices and their product are:

$$\begin{pmatrix} 0.0997 & 0.5356 \\ 0.0204 & 2.6148 \end{pmatrix} \quad \begin{pmatrix} 2.6148 & 0.0204 \\ 0.5356 & 0.0997 \end{pmatrix} \quad \begin{pmatrix} 0.5476 & 0.0646 \\ 1.4538 & 0.2904 \end{pmatrix}$$

The eigenvalues of the product matrix are 0.7512 and 0.0867. ∎

6 Localization phenomenon for CML with a large number of elements

Consider CML with a large number of elements. Suppose that all the maps f_i are identical and correspond to one of the two families considered in Section 4, i.e. the map f has two fixed singular points c and $1 - c$. Consider a sequence of numbers $\bar{w} = \{w_j\}$, where $0 \le w_j \le 1$. We shall assume that

$$\Omega(\bar{w}, i) := \sum_j \gamma_{ij} w_j - w_i \tag{6.1}$$

is positive for any i and any sequence $\bar{w}$, whose elements w_j take only the two values: c or $1 - c$, but there exists a sequence $\bar{w}'$ with only one 0 element, such that $\Omega(\bar{w}', i) < 0$ for some i. The simplest example is a one-dimensional chain with symmetrical coupling and $c < 1/3$.

Theorem 6.1 *Let the graph Γ does not have "free" nodes. Then all SBR measures are localized around the singular points. If there exists a "free" node, then the localization phenomenon does not take place.*

Recall, that the definition of the "free" node was done in Introduction.

According to this statement one can encode all possible SBR measures by binary sequences, taking 0 if the corresponding $x_i \le 1/2$ (actually, close to c) and 1 otherwise (close to $1 - c$). It turns out that all the binary sequences

can appear here, except ones such that for some node all its neighbors (in the graph Γ) have the same value. For example, for the linear cyclic chain the only restriction for the encoding sequence is that among 3 consecutive values there should be both 0 and 1.

If instead of the condition above we assume that

$$\Omega(\bar{w}, i) > 0 \tag{6.2}$$

for any sequence $\bar{w}$ with elements $0, c, 1-c, 1$, then the localization takes place independently on the existence of "free" nodes. Example: a one-dimensional chain with simmetric coupling and $1/3 < c < 1/2$.

Under the last condition CML is "superstable" with respect to various perturbations in the following sense. Consider a localized system and let us change the position in only one component, say x_i. It turns out that the binary code, corresponding to the perturbed map will be changed only in this place, and under the action of the dynamics the code will converge to the original one.

Influence of random noise. Let us take instead of one of the nodes of our network a source of independent random noise. Then the whole system will be stable (unstable) with respect to this noise exactly as in the case of chaotic maps.

References

[1] Blank M.L. Small perturbations of chaotic dynamical systems. Uspekhi Matem. Nauk., 44:6, 1989, p.3-28.

[2] Blank M.L. Chaotic mappings and stochastic Markov chains. Abstracts of Congress IAMP-91, 1992, 6p.

[3] Blank M.L. Singular phenomena in chaotic dynamical systems. Doklady Akad. Nauk (Russia), 328:1(1993), 7-11.

[4] Blank M.L., Keller G. Stochastic stability versus localization in chaotic dynamical systems, in preparation.

[5] Bunimovich L.A., Pesin Ya.G., Sinai Ya.G., Jacobson M.V. Ergodic theory of smooth dynamical systems. Modern problems of mathematics. Fundamental trends. v.2, 1985, P.113-231.

[6] Brichmont J., Kupiainen A. Coupled analytic maps. Nonlinearity, 8:3(1995), 379-396.

[7] Bunimovich L.A., Sinai Ya.G. Spacetime chaos in coupled map lattices. Nonlinearity, 1(1988), 491-516.

[8] Kaneko K. (ed) Theory and Applications of Coupled Map Lattices (New York: Wiley), 1993.

[9] Keller G., Kunzle M. Transfer operators for coupled map lattices. Ergodic Theory & Dyn. Sys., 2(1992), 297-318.

[10] Keller G., Kunzle M. Some phase transitions in coupled map lattices. Physica D, 59(1992), 39-51.

[11] Kunzle M. Invariante Masse fur gekoppelte Abbildungsgitter. Ph.D thesis, Erlangen-Nurnberg University, 1993.

[12] Lasota A., Yorke J.A. On the existence of invariant measures for piecewise monotone transformations, Trans. Amer. Math. Soc., 186(1973),481-488.

[13] Pesin Ya.B., Sinai Ya.G., Space-time chaos in chains of weakly coupled hyperbolic maps. In: Advances in Soviet Mathematics, vol.3, 1991, ed. Ya.G. Sinai, (New-York, Harwood).